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Vibration Analysis: Validation of the Mathematical Model and the Physical Simulink Multibody Model

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Abstract: Partial differential equations (PDEs), matrices, eigenvalues, and eigenvectors are foundational concepts in mathematics and play a critical role in various scientific and engineering applications. PDEs describe the relationship between functions of multiple variables and their partial derivatives, enabling the modeling of complex phenomena such as heat transfer, fluid dynamics, and electromagnetic fields. Eigenvalues and eigenvectors, derived from matrix theory, are essential in understanding the behavior of linear transformations and play a pivotal role in solving systems of differential equations, stability analysis, and in applications like quantum mechanics, structural engineering, and machine learning. Understanding these concepts provides a deeper insight into the structure of systems, allowing for more accurate predictions and optimizations in real-world problems. This study investigates the important role of partial differential equations (PDEs) and matrices in mechanical vibration systems. It presents the modeling of a two-degree-of-freedom system, which includes both rotational and translational motion. The vibration analysis for the given system is conducted using PDEs, matrices, eigenvalues, and eigenvectors. The same analysis, under the same conditions, is then demonstrated through a multibody physical model and non-linear motion equations, with computations performed in Simulink/MATLAB. As a result, the three methods are validated to verify the analysis.

1. Introduction

Vibration analysis plays a particular role in understanding the dynamic behavior of mechanical systems, which is essential for designing efficient and reliable structures and machinery. This study explores vibration analysis by comparing mathematical models with software-based simulations of physical systems. Specifically, it examines how mathematical methods, such as partial differential equations (PDEs), matrices, eigenvalues, and eigenvectors, are used to model and analyze vibrations in mechanical systems. Additionally, the analysis is extended to software simulations using tools like Simulink and MATLAB, which provide a practical approach to verify the accuracy and effectiveness of theoretical models. By validating these two methods, this study aims to highlight the strengths and limitations of both approaches, ensuring a comprehensive understanding of vibration behavior and enhancing the reliability of predictions in real-world applications.

The pendulum-spring-cart system and its dynamics have been widely studied due to its importance in understanding coupled mechanical systems and its applications in engineering. Researchers have extensively explored the behavior, control mechanisms, and energy interactions within such systems, employing both theoretical and experimental approaches.

Vibration analysis is a fundamental aspect of mechanical engineering, essential for ensuring the reliability and performance of dynamic systems. Uncontrolled vibrations can lead to noise, fatigue, and even structural failure [1], [2]. To address these challenges, engineers employ mathematical models, particularly partial differential equations (PDEs), to describe the dynamic behavior of continuous systems such as beams and shafts [3]. In discrete systems, matrix methods, including eigenvalue and eigenvector analysis, are utilized to determine natural frequencies and mode shapes, aiding in the prediction and mitigation of resonance phenomena [4].

Amer et al. investigated the motion of a three degrees-of-freedom (DOF) damped auto-parametric pendulum system, focusing on its stability near resonance conditions. Their work highlighted the intricate interactions and energy exchanges in these systems, which are critical for stability analysis and practical applications in dynamic systems [5].

Anurag et al. studied resonant motion and chaos in a spring-mass-spherical pendulum system, emphasizing the sensitivity of these systems to initial conditions and the occurrence of chaotic behavior. Their findings underscored the need for precise modeling and control to mitigate instability in engineering applications [6].

The use of pendulum-spring systems in robotics and control systems has also been extensively discussed. These setups provide fundamental insights for understanding control algorithms and the dynamics of coupled mechanical systems. Additionally, such systems are used in educational contexts to demonstrate complex dynamic and control concepts [7].

Further studies explored the global dynamics of auto parametric spring-mass-pendulum systems, focusing on modal decoupling and internal resonance phenomena. These studies provided deeper insights into the energy transfer mechanisms and stability considerations in multi-degree-of-freedom systems, contributing to the theoretical understanding and practical design of such systems [8].

Recent advancements in computational tools have facilitated the integration of multibody dynamics (MBD) simulations into vibration analysis. These simulations allow for the modeling of complex systems with multiple degrees of freedom (DOF), capturing both translational and rotational motions [9]. For instance, Nguyen et al. demonstrated the effectiveness of MBD simulations in analyzing the vibrations of mechanical systems, highlighting the method's capability to handle intricate mechanical interactions [10]. Similarly, Kovacs and Ibrahim explored the relationship between MBD computations and nonlinear vibration theory, emphasizing the importance of incorporating system nonlinearities for accurate predictions [11].

The integration of flexible components into MBD models has further enhanced the accuracy of vibration analyses. Shimada and Shabana investigated impact-induced vibrations in flexible multibody systems, employing finite element methods to capture elastic deformations and validate their models experimentally [12]. Moreover, active vibration control strategies have been developed to mitigate unwanted oscillations. Neto et al. proposed a control approach for spatial flexible multibody systems, utilizing piezoelectric materials to achieve effective vibration suppression [13].

These studies collectively enhance the understanding of pendulum-spring-cart systems, providing a foundation for the development of robust and efficient mechanical and control systems in engineering applications. The integration of analytical methods with experimental validations remains critical for advancing the field.

In the context of two-degree-of-freedom (2-DOF) systems, combining rotational and translational motions present unique challenges. Accurate modeling of such systems requires a comprehensive approach that integrates PDEs, matrix algebra, and MBD simulations. This study focuses on developing a mathematical model of a 2-DOF system using PDEs and eigenvalue analysis to determine its natural frequencies and mode shapes. Subsequently, a corresponding multibody physical model is constructed in Simulink/MATLAB to simulate the system's dynamic behavior under identical conditions. By comparing the results from both methods and nonlinear model, the study aims to validate the theoretical model and demonstrate the consistency between analytical and computational approaches.

2. Pendulum-Spring-Cart System

Two-Degree-of-Freedom (2-DOF) system, specifically the pendulum-spring-cart model, is a fundamental mechanical system used to study dynamic behavior. This system is with two independent modes of motion, each corresponding to a distinct degree of freedom. It consists of a pendulum and a cart with a spring between them. The two degrees of freedom correspond to the motion of the cart along a horizontal axis and the angular displacement of the pendulum. This setup presents a coupled dynamic system where the motion of the cart influences the pendulum's motion, and vice versa, leading to complex interactions that are often nonlinear in nature.

The system studied in this work differs from the typical configuration. It consists of a pendulum mounted to the ground, with a spring attached to the pendulum, and a mass connected to the pendulum by the same spring. This setup creates a coupled dynamic system, where the motion of the pendulum affects the motion of the cart and vice versa. A schematic of the system is shown in *figure 1*.

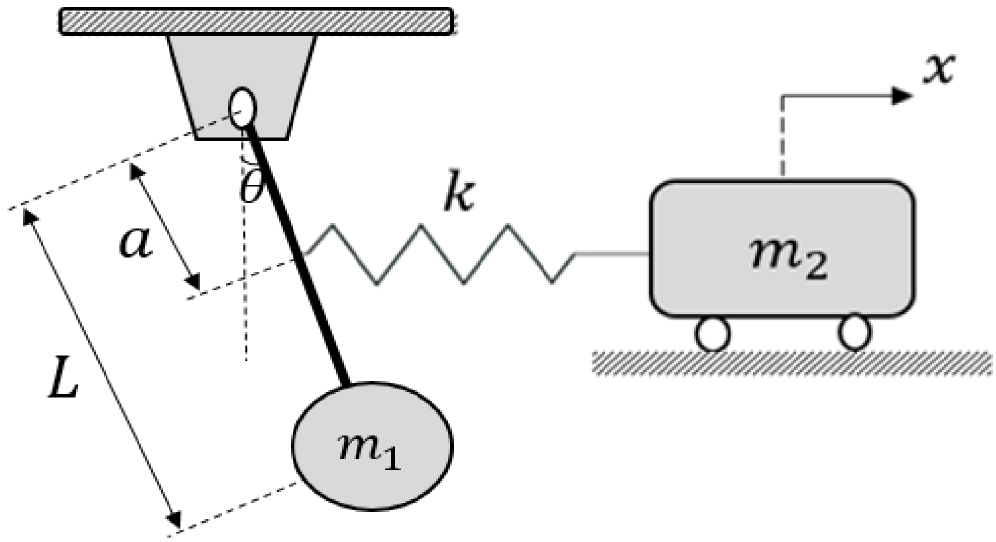

Figure 1. Pendulum-Spring-Cart

The system consists of two different masses and a single spring. The angular position of the pendulum and linear position of the cart are denoted as θ and x, respectively. The parameters a, L, m_1, m_2 and k represent spring connected length, pendulum length, the mass of pendulum sphere, mass of the cart and spring stiffness, respectively. Additionally, the all frictions are considered negligible.

3. Modelling of Pendulum-Spring-Cart

This section presents the mathematical and Simulink models for the system defined in the previous section. The first subsection derives the nonlinear and linear equations of motion for the system using the Lagrange's equations. In the subsequent subsection, the Simulink multibody tool is employed to develop the physical model of the system.

Mathematical Modelling (Equations of Motion)

Lagrange's equations can be stated, for an n-degree-of-freedom system, as shown in *equation 1*.

$$\frac{d}{dt}\left(\frac{\partial(T-U)}{\partial \dot{q}_i}\right)-\frac{\partial(T-U)}{\partial q_i}= Q_i \qquad , i = 1,2,..,n \tag{1}$$

Where, $\dot{q}_i = dq_i/dt$ is the generalized velocity and Q_i is the non-conservative generalized force corresponding to the generalized coordinates q_i. The forces denoted by Q_i may include dissipative (damping) forces or other external forces that cannot be derived from a potential function. Lagrangian energy refers to the difference between the total kinetic energy T and the total potential energy U, of the system.

Nonlinear Model for the System

In the system shown in *Figure 1*, the mass of the cart, m_1, and the moment of inertia of the pendulum, I, contribute to the total kinetic energy, while the springs store potential energy. Additionally, there is potential energy due to the gravitational acceleration, g. The expressions for the total kinetic energy and total potential energy are provided in *equations 2 - 3*, respectively.

$$T = \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}m_2\dot{x}^2 \tag{2}$$

$$U = m_1 g\ (L - L\cos\theta) + \frac{1}{2}k(x - a\sin\theta)^2 \tag{3}$$

By applying the Lagrangian formulation to all generalized coordinates, the equations of motion are derived and presented in *equation 4 - 5*.

$$m_1 L^2 \ddot{\theta} + m_1 g L \sin\theta - ka\cos\theta\, x + \mathrm{k}a^2 \sin\theta\cos\theta = 0 \tag{4}$$

$$m_2\ddot{x} + kx - ka\sin\theta = 0 \tag{5}$$

The system equations are nonlinear due to the presence of trigonometric functions. Nonlinear equations of motion can be written in matrix form as given in *equation 6*.

$$\begin{bmatrix} m_1L^2 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{\theta} \\ \ddot{x} \end{bmatrix} + \begin{bmatrix} m_1gL + ka^2\cos\theta & -ka\cos\theta \\ -ka & k \end{bmatrix}\begin{bmatrix} \sin\theta \\ x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{6}$$

Linear Model for the System

The derived nonlinear equation in the previous subsection can be linearized for small angular displacement, where the approximations $\sin\theta \approx \theta$ and $\cos\theta \approx 1$ based on the assumption of small angular displacement. Subsequently, the equations governing the linearized system are expressed in matrix form in e*quation 7*.

$$\begin{bmatrix} m_1L^2 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{\theta} \\ \ddot{x} \end{bmatrix} + \begin{bmatrix} m_1gL + ka^2 & -ka \\ -ka & k \end{bmatrix}\begin{bmatrix} \theta \\ x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{7}$$

Physical Model for the System (Simulink / MATLAB)

MATLAB/Simulink provides a wide range of essential tools for modeling mechanical, electrical, hydraulic, and other systems. Two key tools for mechanical systems are the Simscape/Foundation Library and Simscape/Multibody.

There are several methods for creating Simulink models for mechanical systems. The first method of them is that the block diagram can be created by using Simscape/Foundation Library/Mechanical which includes machine elements, sensors, actuators and external forces blocks in Simulink/Simscape tool.
One of the other method is that the system can be designed and assembled on a CAD program as SolidWorks. Then, the designed model can be imported to MATLAB/Simulink platform as a Simulink/Multibody model.

The last method –studied in this paper– is that the bodies, springs, sensors, actuators etc., can be designed in Simulink/Multibody tool, directly.

The overall model is presented in *figure 2*. The model comprises two interconnected subsystems: the pendulum subsystem and the cart subsystem, which are depicted in *figures 3* and *4*, respectively. Additionally, the Mechanical Explorer in MATLAB facilitates real-time animation of the model. A snapshot of the initial position configuration is provided in *figure 5*.

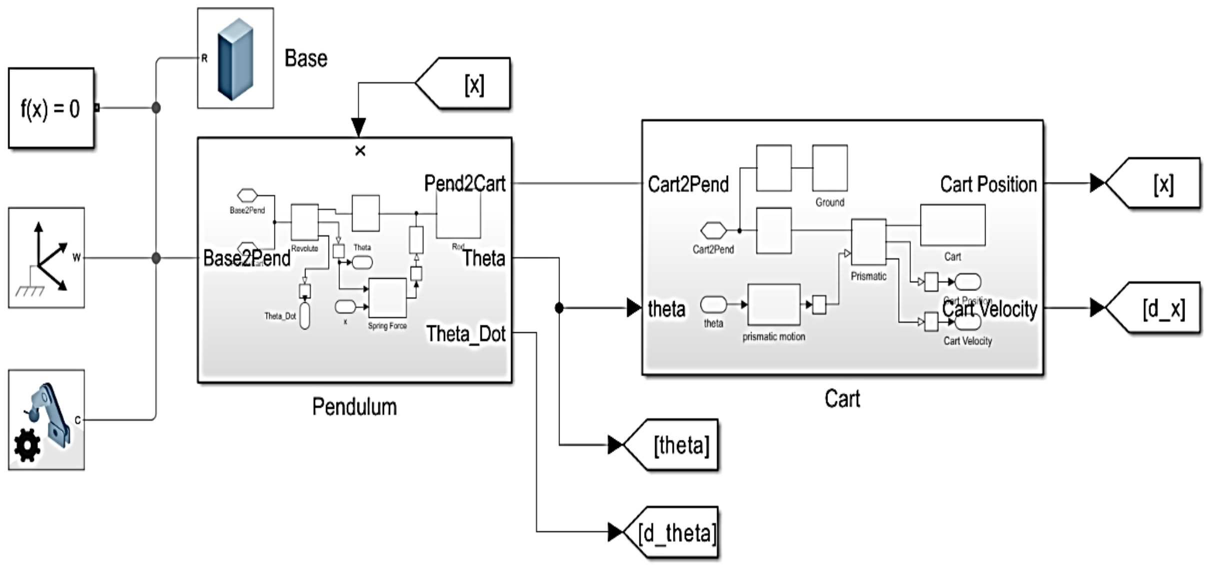

Figure 2. Simulink Model for Overall Pendulum-Spring-Cart System

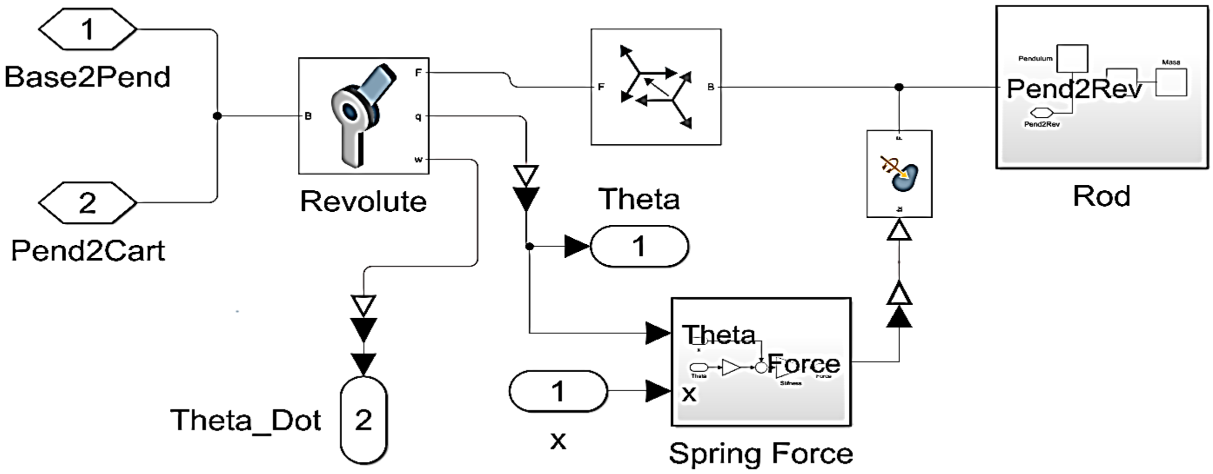

Figure 3. Pendulum Subsystem Block Diagram

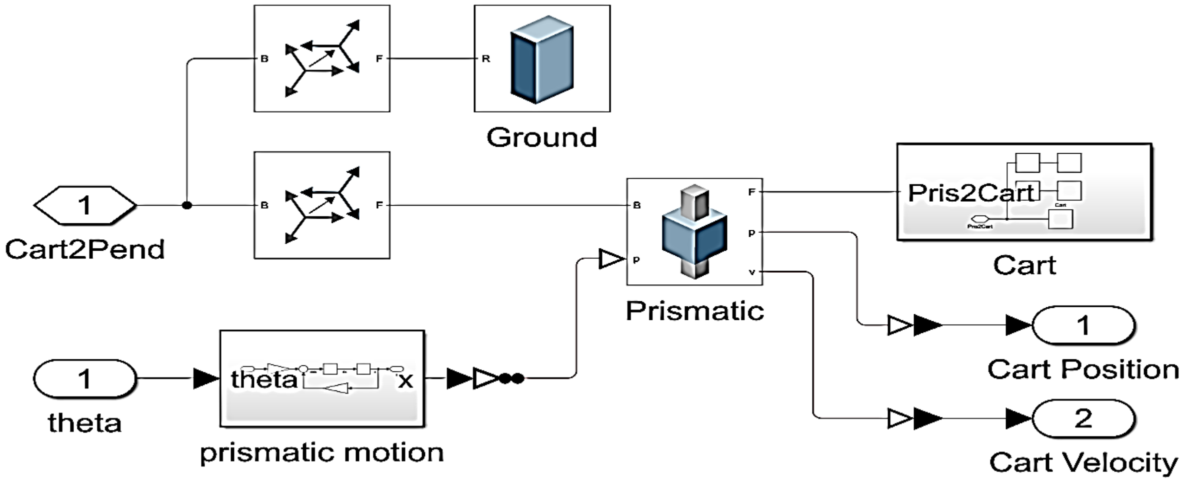

Figure 4. Cart Subsystem Block Diagram

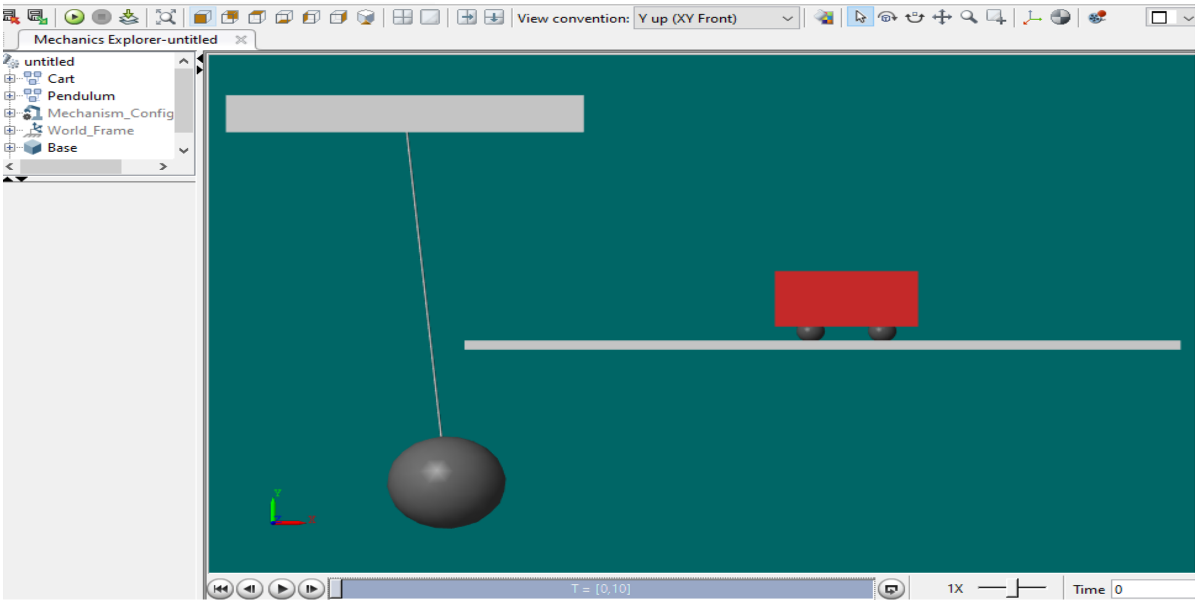

Figure 5. Real-time Animation Window, Initial Position Configuration

4. Natural Frequencies and Mode Shapes

In the present section, three subtitles are mentioned. In the first subsection conducts eigenvalues which refer to natural frequencies. Then, the second subsection provides eigenvectors that refer to mode shapes. Response of variables of the system motion equations are obtained by using eigenvalues and eigenvectors in the last subsection.

Natural Frequencies

The linear equations of motion for the pendulum-spring-cart system were derived in the previous sections. These equations were presented in matrix form in *equation 7*. A simplified version of this equation is provided in *equation 8*.

$$[\underline{\underline{M}}]\,[\underline{\ddot{X}}] + [\underline{\underline{K}}]\,[\underline{X}] = [\underline{0}] \tag{8}$$

I

n this context, $\underline{\underline{M}}$ represents the mass matrix, and $\underline{\underline{K}}$ denotes the stiffness matrix. $\underline{\ddot{X}}$ and $\underline{X}$ are the acceleration and position vectors, respectively. The right-hand side of the equation is equal to zero due to the absence of external forces and torques. Consequently, the vector of external forces, $[\tau \quad F]^T$, is equal to the zero vector, $[\underline{0}]$.

The homogeneous second-order linear partial differential equations (PDEs) have constant coefficients. The roots of the motion equations, which are also referred to as the eigenvalues of the system, determine the solution behavior. The general solution for $\underline{X}(\mathrm{t})$ and its second derivative can be expressed as shown in *equation 9*.

$$\underline{X}(t) = Ae^{iw_n t} \text{ and } \underline{\ddot{X}}(t) = -Aw_n{}^2 e^{iw_n t} \tag{9}$$

Here, A represents the amplitude, *i* is imaginary part and t denotes time. By substituting *equation 9* into *equation 8*, the *equation 10* is attained. It provides the responses of the position vector, eigenvalues, and natural frequencies, w_n.

$$\left(-w_n{}^2\left[\underline{\underline{M}}\right] + \left[\underline{\underline{K}}\right]\right)Ae^{iw_n t} = [\underline{0}] \quad , \quad \left[\underline{\underline{K}} - w_n{}^2\underline{\underline{M}}\right] = [\underline{0}] \tag{10}$$

To obtain characteristic equation, determinant of the *equation 10*, $det(\left[\underline{\underline{K}} - w_n{}^2\underline{\underline{M}}\right])$ should be zero. If physical parameters, *4 kg of pendulum's mass*, *6 kg of cart's mass, 100 N/m of spring stiffness*, *0.4 meter of the length of pendulum* and *0.2 meter the distance between revolute joint and spring-pendulum connection point* is used during taking determinant, resulting characteristic equation is obtained as given below in *equation 11*.

$$3.84\lambda_i{}^2 - 182.18\lambda_i + 1569.60 = 0 \tag{11}$$

Also eigenvalues of the determinant function provides natural frequencies. Therefore, a relation equation between eigenvalues and natural frequencies can be given as $\lambda_i = w_{ni}{}^2$. However, the solution of the *equation 11* provides eigenvalues, $\lambda_1 = 11.316$ and$\lambda_2 = 36.128$. Natural frequencies for two generalized coordinates are $w_{n1} = 3.364\ Hz$, $w_{n2} = 6.011\ Hz$.

Mode Shapes

There are two modes for each natural frequency. To determine eigenvectors, the calculated natural frequencies substituted into *equation 10* and *equation 12* is obtained.

$$\begin{bmatrix}(m_1 gL + ka^2 - m_1 L^2 w_{ni}{}^2)\theta - kax \\ (k - m_2 w_{ni}{}^2)x - ka\theta\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \tag{12}$$

Mode of the first natural frequency, (3.364 *Hz*)*:*

After using parameters into *equation 12*, eigenvector or mode for the first natural frequency is given in the *equation 13*.

$$\begin{bmatrix}x - 0.623\theta \\ x - 0.623\theta\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}, \ A^{(1)} = \begin{Bmatrix}A^{(1)}\cos\left(w_{n1}t + \emptyset_1\right) \\ 0.623A^{(1)}\cos\left(w_{n1}t + \emptyset_1\right)\end{Bmatrix} \tag{13}$$

Mode of the second natural frequency, (6.011 *Hz*)*:*

The same procedure to find first mode is utilized to determine the second mode. The mode equation and eigenvector are given below in *equation 14*.

$$\begin{bmatrix}x + 0.171\theta \\ x + 0.171\theta\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}, \ A^{(2)} = \begin{Bmatrix}A^{(2)}\cos\left(w_{n2}t + \emptyset_2\right) \\ -0.171A^{(2)}\cos\left(w_{n2}t + \emptyset_2\right)\end{Bmatrix} \tag{14}$$

Where, $A^{(1)}$ and $\emptyset_1$ are mode and phase for first nature frequency while $A^{(2)}$ and $\emptyset_2$ denotes mode and phase for second natural frequency. Mode shapes for both frequencies are given in *figure 6*.

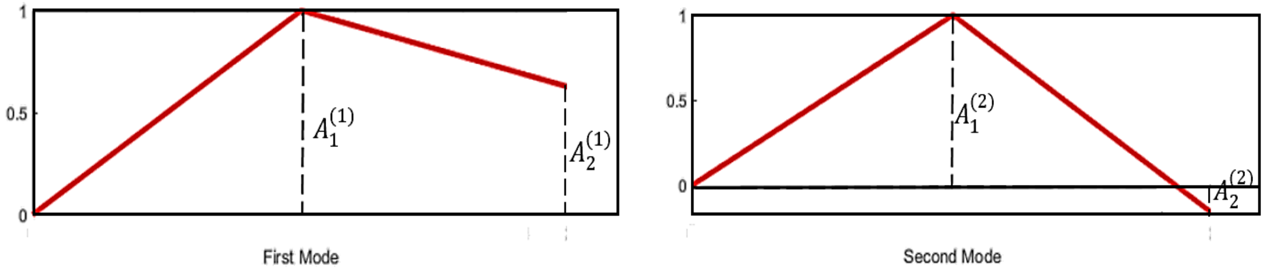

Figure 6. First and Second Mode Shapes

Solution of Motion Equations

In previous two subsections, eigenvalues and eigenvectors are determined. Therefore, angular and linear positions can written by using natural frequencies and amplitudes which are refer to eigenvalues and eigenvectors, respectively. The position vector is given below in *equation 15*.

$$\underline{X}(t) = \begin{bmatrix} A^{(1)}\cos(w_{n1}t + \emptyset_2) + A^{(2)}\cos(w_{n2}t + \emptyset_2) \\ 0.623A^{(1)}\cos(w_{n1}t + \emptyset_1) - 0.171A^{(2)}\cos\,(w_{n2}t + \emptyset_2) \end{bmatrix} \tag{15}$$

The velocity vector is obtained if derivative of position vector with respect to time is taken. Therefore, four equations have been in total which included two position equations and two velocity equations. They provide solutions of four unknown parameters which are $\emptyset_1$, $\emptyset_2$, $A^{(1)}$,and $A^{(2)}$. By using initial conditions for position vector of $[5^{\circ}\ 0.02m]^T$ and velocity vector of $[0\ 0]^T$, phases of $\emptyset_1$ and $\emptyset_2$ are determined of zeros while modes for both natural frequencies equal to each other. Their values equal to *0.0440*. The resultant equations of positions and velocities are presented in *equation 16*.

$$\begin{bmatrix} \theta(t) \\ x(t) \\ \dot{\theta}(t) \\ \dot{x}(t) \end{bmatrix} = \begin{bmatrix} 0.440\cos(3.364t) + 0.440\cos(6.011t) \\ 0.027\cos(3.364t) - 0.0074\cos(6.011t) \\ -0.1480\sin(3.364t) - 0.2645\sin(6.011t) \\ -0.091\sin(3.364t) + 0.045\sin(6.011t) \end{bmatrix} \tag{16}$$

5. Results

In the previous sections, nonlinear motion equations, linear motion equations, their natural frequencies, modes, eigenvalues, eigenvectors, linear equations responses were conducted profoundly. Then, a software physical model of the defined system was presented. In this model, links, translation elements, joints and sensors are utilized with their physical units. The physical parameters as mass, inertia, lengths, stiffness, etc., are written in the Simulink blocks.

Same parameters and equal initial values are used for linear, nonlinear and physical models. The initial values for angular and linear velocities are zero while initial angular and linear displacement are 5° and 0.02 meter, respectively.

Here, two subsection are organized to present the responses of angular displacement, linear displacement, angular velocity and linear velocity for linear, nonlinear and physical models. In the first subsection, the positions' responses are presented. Then, following subsection provides to see validation for velocities responses.

Position Analysis for Pendulum – Spring – Cart

There are two displacements studied for the Pendulum-Spring-Cart system. Both angular displacement of pendulum and linear position of cart are analyzed. Pendulum's angular displacement for three mentioned models with respect to time is given in same *figure 7* while cart's linear position for three model are illustrated in *figure 8*. The curves of physical model's output, nonlinear model's and linear model's output are shaped with solid black line, dashed blue line and center red line.

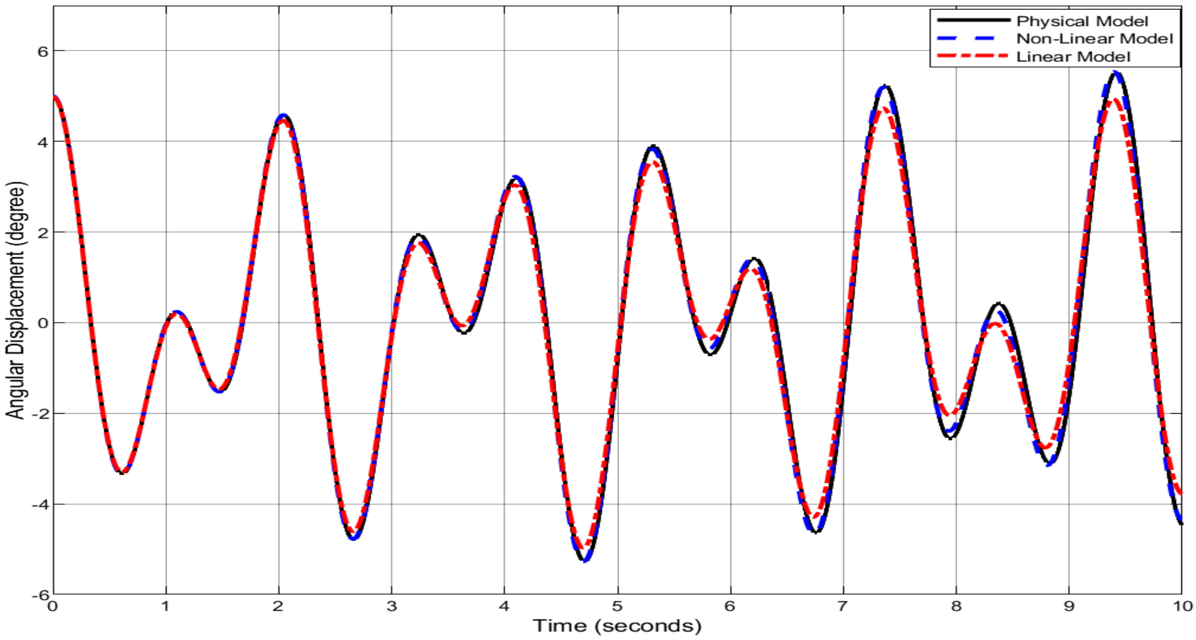

Figure 7. Angular Position of Pendulum

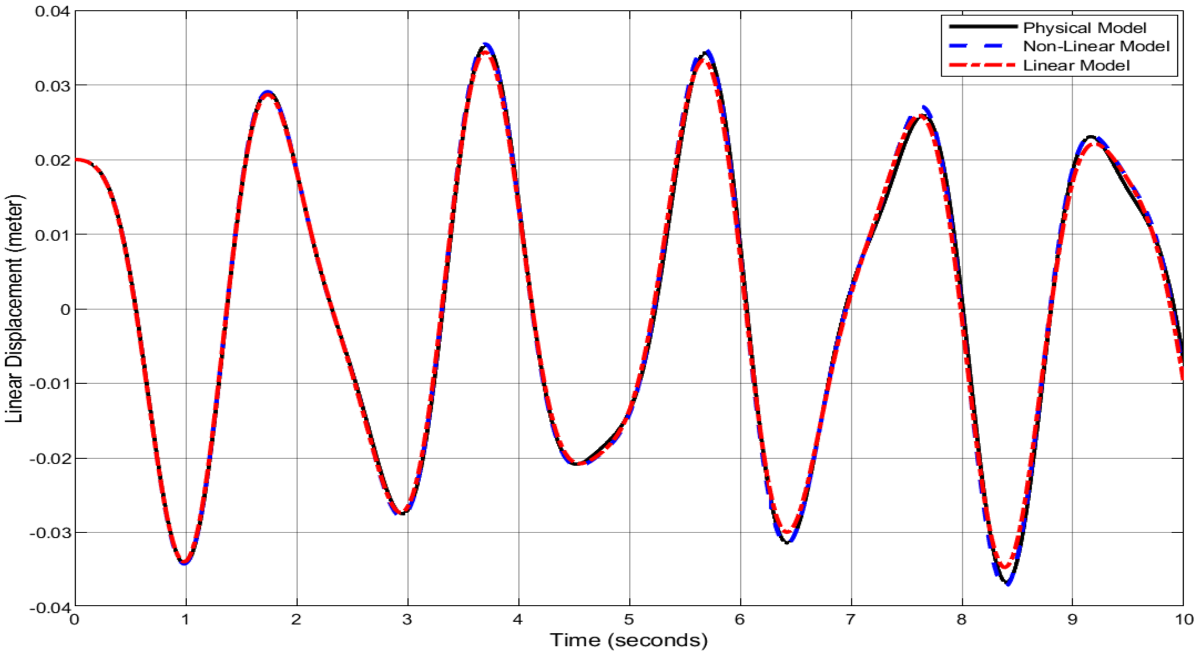

Figure 8. Linear Position of Cart

Velocity Analysis for Pendulum – Spring – Cart

Both angular velocity of pendulum and linear velocity of cart are analyzed. Pendulum's angular velocity for three mentioned models with respect to time is given in same *figure 9* while cart's linear velocity for three models are illustrated in *figure 10*. The curves of physical model's output, nonlinear model's and linear model's output are drawn with solid black line, dashed blue line and center red line.

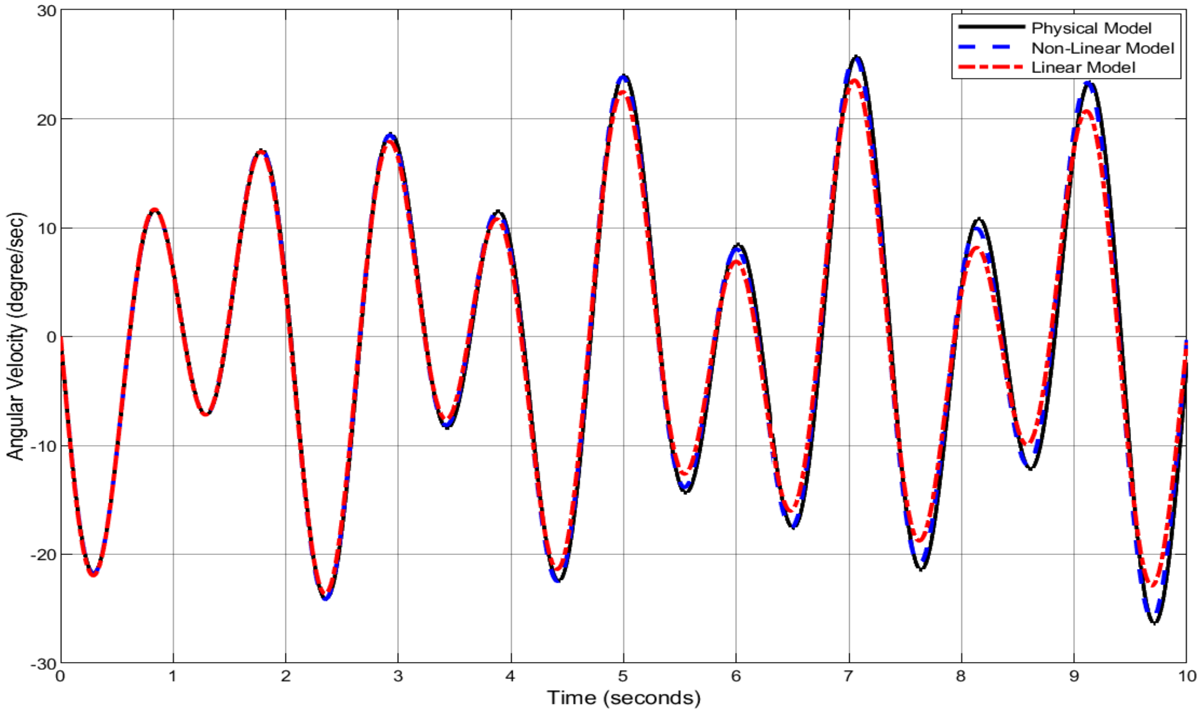

Figure 9. Angular Velocity of Pendulum

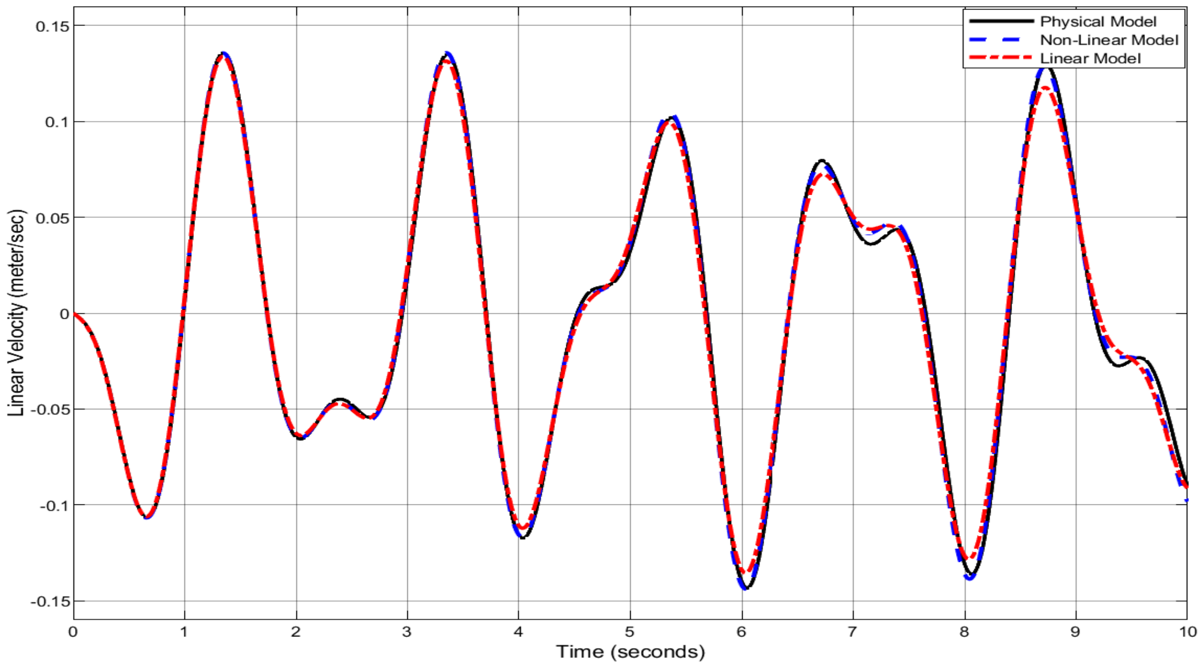

Figure 10. Linear Velocity of Cart

6. Conclusion and Discussion

This study has presented an analysis of the vibration behavior in a pendulum-spring-cart system using mathematical models and physical simulations. By solving the equations of motion for both nonlinear and linear cases, the theoretical results were compared with simulations created in MATLAB/Simulink. Important values like natural frequencies, mode shapes, and system responses were calculated and analyzed. The results show that both methods are effective for studying and predicting the dynamics of mechanical systems when the same initial conditions and parameters are used.

The validation shows a strong agreement between the linear, nonlinear, and physical models. Small differences were found, mainly because of the simplifications in the linearized model and simulation accuracy. However, these differences are small enough to confirm the reliability of the methods used in this study.

The results highlight the importance of using different methods to study vibrations in mechanical systems. Mathematical modeling, based on Lagrange's equations, gives a clear theoretical understanding of the system. At the same time, physical simulation models help with practical insights and visual understanding, which are useful for real-world applications.

One key point in the analysis was the validation of natural frequencies and mode shapes, which are important for checking system stability and response. The good match between mathematical and simulation results shows that the chosen methods and parameters are reliable. However, the linearized model's use of small-angle approximations limits its accuracy for large displacements, where nonlinear effects become more important.

Future studies could include damping and external forces to make the analysis more realistic. Also, expanding the physical model to study three-dimensional movements or different kinds of forces could give more detailed results. Researchers could also explore optimization methods to improve system performance for applications like robots or vibration control systems.

In conclusion, this study validates the use of mathematical and simulation methods in vibration analysis, showing their strengths and suggesting areas for further research.

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