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On the Relation β_{ss}^{*} in Module Theory

Modül Teoride β_{ss}^{*} Bağıntısının Temel Özellikleri

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ABSTRACT

In this study, some features of the Goldie ss-lifting modules concept and generalizations of this module class are given with the help of the relation β_{ss}^{*} defined in the article (Gömleksiz & Nişancı Türkmen, 2023). In this relation $X\beta_{ss}^{*}Y$, which is defined as submodules X and Y of the module H provided that $\frac{X+Y}{X} \subseteq \frac{Soc_s(H)+X}{X}$ and $\frac{X+Y}{Y} \subseteq \frac{Soc_s(H)+Y}{Y}$ is determined by conditions. The important properties of this relation in Goldie-ss-lifting modules have been studied. In this article, Goldie-ss-supplemented modules are considered as a special case of Goldie*-supplemented modules and Goldie-ss-lifting modules are considered as a special case of Goldie*-lifting modules, and the basic module structure theorems are included with the help of the relation β_{ss}^{*}, which is more special than the relation β^{*}. The classification of the ss-semi-local modules was made using the relation β_{ss}^{*}. It has been proved that the factor modules of Goldie-ss-supplemented modules are also Goldie-ss-supplemented modules. It has been shown that $S_{\beta_{ss}^{*}}(H)$, which is the set of equivalence classes according to the relation β_{ss}^{*} for submodules of a module H, has a monoid structure. With the help of fully invariant submodules, it has been shown that every direct summand of a Goldie-ss-supplemented module is a Goldie-ss-supplemented module. In addition, ss-supplemented modules and Goldie-ss-supplemented modules classes, and ss-lifting modules and Goldie-ss-lifting modules classes were compared.

Keywords: The Relation β_{ss}^{*}, Goldie-ss-Lifting Module, Goldie-ss-Supplemented Module

ÖZET

Bu çalışmada, Goldie ss-yükseltilebilir modüller kavramının birtakım özellikleri ve bu modül sınıfının genellemeleri (Gömleksiz ve Nişancı Türkmen, 2023) adlı makalede tanımlanan β_{ss}^{*} bağıntısı yardımıyla verilmiştir. Bu bağıntıda $X\beta_{ss}^{*}Y$ olarak tanımlanan H modülünün X ve Y alt modülleri için $\frac{X+Y}{X} \subseteq \frac{Des_s(H)+X}{X}$ ve $\frac{X+Y}{Y} \subseteq \frac{Des_s(H)+Y}{Y}$ koşulları ile belirlenmiştir. Bu bağıntının Goldie-ss-yükseltilebilir modüllerdeki önemli özellikleri incelenmiştir. Bu makalede, Goldie-ss-tümlenmiş modüller Goldie*-tümlenmiş modüllerin bir özel hali ve Goldie-ss-yükseltilebilir modülleri de Goldie*-yükseltilebilir modüllerin bir özel hali olarak ele alınarak β^{*} bağıntısından daha özel olan β_{ss}^{*} bağıntısı yardımıyla temel modül yapı teoremlerine yer verilmiştir. Ss-yarıyerel modüllerin β_{ss}^{*} bağıntısı kullanılarak sınıflandırılması yapılmıştır. Goldie-ss-tümlenmiş modüllerin bölüm modüllerinin de Goldie-ss-tümlenmiş modüller olduğu ispatlanmıştır. Bir H modülünün alt modülleri için β_{ss}^{*} bağıntısına göre denklik sınıflarının kümesi olan $S_{\beta_{ss}^{*}}(H)$ ının bir monoid yapısına sahip olduğu gösterilmiştir. Karakteristik alt modüller yardımıyla Goldie-ss-tümlenmiş bir modülün her direkt toplam teriminin Goldie-ss-tümlenmiş modül olduğu gösterilmiştir.

Ayrıca ss-tümlenmiş modüller ile Goldie-ss-tümlenmiş modüllerin sınıfları, ve ss-yükseltilebilir modüller ile Goldie-ss-yükseltilebilir modüllerin sınıfları mukayese edilmiştir.

Anahtar kelimeler: β^*_{ss} bağıntısı, Goldie-ss-Yükseltilebilir Modül, Goldie-ss-Tümlenmiş Modül

1. INTRODUCTION

In this work, R denotes as an associative ring with unit element 1_R, and the modules situated unitary left R-modules. For a module H, we use $N \leq H$ to denote N is a submodule of H. $Rad(H)$ will denote the Jacobson radical of H. A submodule $K \subseteq H$ is said to be *small* in H (denoted by the $K \ll H$) if $N + K \neq H$ for any proper submodule N of H. Lifting modules and variations were studied by many authors (see (Clark et. al., 2006), (Eryılmaz, 2021). Following (Clark et. al., 2006, we call a module H *lifting* if for each $N \subseteq H$ there is a direct summand K of H so that $K \subseteq N$ and $N/K \ll H/K$. A module M is said to be *supplemented* if for each $N \subseteq H$, there is $K \subseteq H$ so that $H = N + K$ and $N \cap K \ll K$ (in this stage K is a supplement of N in H). A module H is said to be *semilocal* if for every submodule N of H, there is a $L \subseteq H$ such that $N + L$ is equal to H and $N \cap L \ll H$. According to (Eryılmaz, 2020), a module H is said to be *ss-lifting* if for each $N \subseteq H$ there is a direct summand K of H such that $K \subseteq N$ and $N/K \subseteq Soc_s(H/K)$, where $Soc_s(X)$ is the sum of all simple submodules which are small in a module X. A module H is said to be *ss-supplemented* if its submodules have an ss-supplement in H. Let $K, L \leq H$. K is said to be a *(weak) ss-supplement* of L in H provided that $N + K{=}H$ and $N \cap K \subseteq Soc_s(H)$ (Kaynar et.al., 2020). A module H is said to be *ss-semilocal* if its submodules have a weak ss-supplement in H (Olgun & Türkmen, 2020).

By $S(H)$ we denote the set of all submodules of a module H. Then $X, Y \in S(H)$ over the set $S(H)$ relation β^* is defined as follows. "$X\beta^*Y \Leftrightarrow (X + Y)/Y \ll H/Y)$ and $(X + Y)/X \ll H/X$". Let H be a module. If for each submodule $X \subseteq H$ $X\beta^*Y$ thus, if a direct summand D of H can be found, then the module H, Goldie*-lifting. Let H be a module. If for each $X \subseteq H$ submodule there will be a supplement S of X such that $X\beta^*S$, then H is said to be *Goldie*-supplemented*.

In (Gömleksiz & Nişancı Türkmen, 2023), it is defined Goldie-ss-supplemented and Goldie-ss-lifting modules based on the definition of the relation β^*_{ss}. Let X, $Y \subseteq H$ be modules. It is said to be *X and Y are equivalent by the relation β^*_{ss}*, $X\beta^*_{ss}Y$ providing the conditions the $\frac{X+Y}{X} \subseteq \frac{Soc_s(H)+X}{X}$ and $\frac{X+Y}{Y} \subseteq \frac{Soc_s(H)+Y}{Y}$ for X and Y submodules taken in the set of submodules of H. A module H is said to be *Goldie-ss-supplemented* if for each $\subseteq H$, there is a submodule S of H with $N\beta^*_{ss}S$, where S is an ss-supplement in H. A module H is called *Goldie-ss-lifting* (resp., (P^*_{ss})) if for any $N \subseteq H$, there exists an ss-supplement submodule (a direct summand) D of H so that $N\beta^*_{ss}D$ $(resp., \frac{N}{D} \subseteq Soc_s\left(\frac{H}{D}\right), D \leq N)$. Clearly every (P^*_{ss})-module is Goldie-ss-lifting and every Goldie-ss-lifting module is Goldie-ss-supplemented.

2. MATERIALS AND METHODS

In this section, the main features of the relation β^*_{ss} that we will give in the main part of our study will be given by quoting from the reference (Gömleksiz & Nişancı Türkmen, 2023).

Theorem 2.1. Let H be a module and $X, Y \subseteq H$. In this case, the following expressions are equivalent.

(1) $X\beta^*_{ss}Y$,

(2) There are equations $X + A = H$ and $Y + A = H$ for each submodule of the module H satisfying the condition $X + Y + A$,

(3) Let X and Y be semisimple submodules of H be given, $Y + K = H$ for every $K \subseteq H$ satisfying the conditions the sum $X + K$ is H. Also the sum $X + Z$ is H for every $Z \subseteq H$ satisfying the condition $Y + Z = H$.

Proof. (1) ⇒ (2) Let $X\beta_{ss}^{*}Y$. Let's take any submodule A of H with $X + Y + A = H$. It follows from $X + Y + A = H$ that $(X + Y)/Y + (A + Y)/Y$ is H/Y. By the hypothesis, $(X + Y)/Y \subseteq Soc_s(H/Y)$. So we get $Y + A = H$ is found. Similarly, it can be shown that $X + A = H$.

(2)⇒(3) Let's take any submodule $K \subseteq H$ satisfying the condition $X + K = H$. Then $X + K + Y = H$ can be written and $Y + K = H$ is found from the hypothesis. Similarly let's take any submodule $L \subseteq H$ that satisfies the condition $Y + L = H$. Here $X + L + Y = H$ can be written and again $X + L = H$ is found from the hypothesis.

(3) ⇒(2) Let's take any submodule A of H that satisfies the condition $X + Y + A = H$. If $X + (Y + A) = H$ is taken as (3) $Y + (Y + A) = H$ can be written. From here, the result is $Y + A = H$. If the roles of X and Y are changed in the steps, it can be shown that $X + A = H$ similarly.

(3)⇒ (1) It follows from $(X + Y)/Y \subseteq Soc_s(H/Y)$ and $(X + Y)/X \subseteq Soc_s(H/X)$ that $(X + Y)/Y \ll H/Y$ and $(X + Y)/X \ll H/X$. Since submodules X and Y are semisimple, factor modules $(X + Y)/Y$, $(X + Y)/X$ are semisimple by 8.1.5. Corollary 2 in (Kasch, 1982). Let's take the submodule $B/Y \subseteq H/Y$ with $(X + Y)/Y + B/Y = H/Y$. In this case $X + Y + B = H$ and since $Y \subseteq B$, we have the sum $X + B$ is H. From hypothesis, the sum $Y + B$ is H. As $Y \subseteq B$, we obtain that $B = H$. Thus, the result $(X + Y)/Y \ll H/Y$ is reached. It can also be shown to be $(X + Y)/X \ll H/X$ with similar operations. Thus $(X + Y)/Y \subseteq Soc_s(H/Y)$ and $(X + Y)/X \subseteq Soc_s(H/X)$ are obtained.

Example 2.2. (i) Two modules isomorphic to each other in any S-module β_{ss}^{*} may not be equivalent according to the relation. For example, $S = \left\{\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \middle| a, b, c \in F\right\}$, where F is field. Let's take ${}_{S}S$. Then submodules $X = \begin{bmatrix} 0 & F \\ 0 & 0 \end{bmatrix}$ and $Y = \begin{bmatrix} 0 & 0 \\ 0 & F \end{bmatrix}$ are isomorphic. But they are not equivalent according to the relation β_{ss}^{*}.

(ii) Given ${}_{\mathbb{Z}}\mathbb{Z} \times \mathbb{Z}$. Therefore $\mathbb{Z} \times \{0\}$ and $\{0\} \times \mathbb{Z}$ are also isomorphic to each other. On the other hand, they are not equivalent according to the relation β_{ss}^{*}.

Corollary 2.3. The set of small semisimple submodules in a module H form a unique equivalence class according to the relation β_{ss}^{*}.

Theorem 2.4. Let $X, Y \subseteq H$ be modules. If $X\beta_{ss}^{*}Y$, then submodules X and Y of H are ss-supplement of each other.

Proof. Suppose the submodule $U \subseteq H$ is an ss-supplement of X. Since $X\beta_{ss}^{*}Y$, then $Y + U = H$ according to Theorem 2.1. It can be shown that the submodule $U \subseteq H$ is an bir ss-supplement of Y, let's take a submodule $V \subseteq U$ with $Y + V = H$. In this case, since $X\beta_{ss}^{*}$, then $X + V = H$ according to Theorem 2.1. From here, since the submodule U is an ss-supplement of X and $V \subseteq U$, then we have $X \cap U$ is semisimple by 8.1.5.Corollary 1 in (Kasch, 1982). It follows that $X \cap V$ is semisimple.

Therefore the submodule U is also an ss-supplement of Y. Also since the relation β_{ss}^{*} is symmetric, similarly the ss-supplemented of Y is also the ss-supplemented of X.

Theorem 2.5. Let X and Y be semisimple submodules of a module H with $J \subseteq Soc_s(H)$. Then $X\beta_{ss}^{*}Y \Leftrightarrow X\beta_{ss}^{*}(Y+J)$.

Proof. $(\Rightarrow)$: Let's assume that $X\beta_{ss}^{*}Y$. Let's take any submodule $K \subseteq M$ satisfying $H = X + (Y+J) + K$. Since $H = X + (Y+J) + K$ and $J \subseteq Soc_s(H)$, the equality $H = X + Y + K$ is obtained. Consider that $H = X + Y + K$ and $X\beta_{ss}^{*}Y$ are obtained Theorem 2.1. $H = X + K$ and $H = Y + K$ are obtained. From here, the result of $H = X + K$ and $H = (Y+J) + K$ is reached. Again, according to Theorem 2.1., $X\beta_{ss}^{*}(Y+J)$ is found.

$(\Leftarrow)$: Let's assume that $X\beta_{ss}^{*}(Y+J)$. Let's take any submodule $K \subseteq H$ satisfying $H = X + Y + K$. In this case $H = X + (Y+J) + K$ can be written. Also, since $X\beta_{ss}^{*}(Y+J)$ according to Theorem 2.1. $H = X + K$ and $H = (Y+J) + K$ are found. By acception $J \subseteq Soc_s(H)$, the result is $H = Y + K$. Then $X\beta_{ss}^{*}Y$ is obtained according to Theorem 2.1.

Theorem 2.6. Let H be a module with $Soc_s(H) = \{0\}$. If there is a decomposition $H = H_1 \oplus H_2$ for the submodule H_1 of H and the semisimple submodule H_2 of H, then $X \subseteq H$ and $X\beta_{ss}^{*}H_1$ with $H = H_2 \oplus X$.

Proof. Let's assume is $X\beta_{ss}^{*}H_1$. From the equation $H = H_1 \oplus H_2$, the submodule H_2 is an ss-supplement of H_1. Since $X\beta_{ss}^{*}H_1$, the semisimple submodule H_2 is also an ss-supplement of X according to Theorem 2.4. So $H = H_2 + X$ and $H_2 \cap X \ll M_2$ can be written. Since H_2 semisimple, so is $H_2 \cap X$. Hence is $H_2 \cap X \subseteq Soc_s(H)$. By the hypothesis $Soc_s(H) = \{0\}$, $H_2 \cap X \subseteq Soc_s(H)$ $=\{0\}$ with $H = H_2 \oplus X$.

Theorem 2.7. Let S be a semisimple submodule of an R-module H and let I be an ideal of R. Then $IS\beta_{ss}^{*}I^nS$ for every $n \in \mathbb{Z}^+$. In addition, for every $n \in \mathbb{Z}^+$, it satisfied that $I\beta_{ss}^{*}I^n$.

Proof. Let's apply the proof by induction on n so that $n = 1$. Then $IS\beta_{ss}^{*}IS$ is provided since β_{ss}^{*} is the reflexive relation. Let us now assume that the claim is true for $n > 1$. Suppose we take $B \subseteq H$ such that $IS/I^nS + B/I^nS = H/I^nS$. In this case $IS + B = H$. From here, there are the equations $I^2S + IB = IH, \ldots, I^nS + I^{n-1}B = I^{n-1}H$. Using these equations, $H = IS + B \subseteq IH + B = I^2S + IB + B \subseteq I^2H + IB + B = I^3S + I^2B + IB + B \subseteq \cdots \subseteq I^nS + I^{n-1}B + \cdots + IB + B \subseteq I^nS + B \subseteq H$. Thus $B/I^nS = H/I^nS$. Then we have $(IS + I^nS)/I^nS \ll H/I^n$S and $(IS + I^nS)/IS = \{0_{H/IS}\} \ll H/IS$. Also $(IS + I^nS)/I^nS$ is semisimple, since S is semisimple according to the (Kasch, 1982) in Corollary 3. So since $IS + I^nS/IS \subseteq Soc_s(H/IS)$ and $IS + I^nS/I^nS \subseteq Soc_s(H/I^nS)$ is $IS\beta_{ss}^{*}I^nS$.

3. RESULTS AND DISCUSSION

Proposition 3.1. Given a ss-semilocal module H. Then its semisimple submodule are equivalent to an ss-semilocal submodules of H by the relation β_{ss}^{*}.

Proof. Let H be an ss-semilocal module. Let's take any semisimple submodule X of H. Since H is ss-semilocal, there is a submodule $Z \subseteq H$ so that $X + Z = h$ and $X \cap Z \subseteq Soc_s(H)$. Here X is also a weak ss-supplement of Z. Also $X\beta_{ss}^{*}X$ can be written since the relation β_{ss}^{*} has reflexive property. As a result, every semisimple submodule of h is equivalent to an ss-semilocal submodule of H .

Lemma 3.2. (See (Gömleksiz & Nişancı Türkmen, 2023)) Let $X_1, X_2, Y_1, Y_2 \subseteq h$, such that $X_1\beta^*_{ss}Y_1$ and $X_2\beta^*_{ss}Y_2$. Then $(X_1 + X_2)\beta^*_{ss}(Y_1 + Y_2)$ and $(X_1 + Y_2)\beta^*_{ss}(Y_1 + X_2)$.

Proof. Suppose that $X_1\beta^*_{ss}Y_1$ and $X_2\beta^*_{ss}Y_2$. Then $X_1 + Y_1 \subseteq Soc_s(H) + X_1$, $X_1 + Y_1 \subseteq Soc_s(H) + Y_1$, $X_2 + Y_2 \subseteq Soc_s(H) + X_2$ and $X_2 + Y_2 \subseteq Soc_s(H) + Y_2$. Hence by using above inclusions, we can easily see that $(X_1 + X_2)\beta^*_{ss}(Y_1 + Y_2)$ and $(X_1 + Y_2)\beta^*_{ss}(Y_1 + X_2)$.

Corollary 3.3. Let H be a module. If a least one small semisimple submodule L of H which satisfies the equations $X + L = Z + L = X + Z$ for every ss-semilocal semisimple submodule X of H and a ss-semilocal Z of H, then $X\beta^*_{ss}(X + Z)$ and $Z\beta^*_{ss}(Z + X)$.

Proof. Let's assume that module H is ss-semilocal. By Proposition 3.1, a module H has a weak ss-supplement Z so that $\forall X \subseteq H$ is $X\beta^*_{ss}Z$ for the semisimple submodule. In this case, a submodule $W \subseteq H$ can be found such that $Z + W = H$ and $Z \cap W \subseteq Soc_s(H)$. According to Lemma 3.2, we have $X\beta^*_{ss}(X + Z)$ and $Z\beta^*_{ss}(Z + X)$.

Remark 3.4. In Lemma 3.2, the finite sum is extensible, but infinite sum cannot be extended. For example, let's take $R = \mathbb{Z}$ and $H = \mathbb{Q}$. We know that $Soc_s(\mathbb{Q}) = \mathbb{Q} = \sum_{m\in\mathbb{Z}^+} \mathbb{Z}\frac{1}{m}$. Here $\mathbb{Z}\frac{1}{m}\beta^*_{ss}\{0\}$, since $\mathbb{Z}\frac{1}{m} \subseteq Soc_s(\mathbb{Q})$ for every $m \in \mathbb{Z}^+$. If Lemma 3.2 was satisfied in countable sums, then $\mathbb{Q} = \sum_{m\in\mathbb{Z}^+} \mathbb{Z}\frac{1}{m}\beta^*_{ss}\{0\}$. This would introduce the contradiction $\mathbb{Q} \subseteq Soc_s(\mathbb{Q})$.

Definition 3.5. Let H be a module. According to the relation β^*_{ss}, the submodule of H let's denote the set of equivalence classes with $S_{\beta^*_{ss}}(H)$. That is $S_{\beta^*_{ss}}(H) = \{\bar{X}_{\beta^*_{ss}} | X \subseteq H\}$ where is $\bar{X}_{\beta^*_{ss}} = \{Y | X\beta^*_{ss}Y\}$.

Theorem 3.6. $S_{\beta^*_{ss}}(H)$ has a monoid structure.

Proof. It is clear that a binary operation $+: S_{\beta^*_{ss}}(H) \times S_{\beta^*_{ss}}(H) \to S_{\beta^*_{ss}}(H)$ is defined by $(\bar{X}_{\beta^*_{ss}}, \bar{Y}_{\beta^*_{ss}}) \in S_{\beta^*_{ss}}(H) \times S_{\beta^*_{ss}}(H)$ with $\bar{X}_{\beta^*_{ss}} + \bar{Y}_{\beta^*_{ss}} = (\overline{X + Y})_{\beta^*_{ss}}$. Then + has the associative property. So $\bar{0}_{\beta^*_{ss}} \in S_{\beta^*_{ss}}(H)$ is the unitary element according to this operation. So $S_{\beta^*_{ss}}(H)$ has a monoid structure.

Lemma 3.7. Given the decomposition of modules $K = A \oplus B$. For $X, S \subseteq A$, if $S\beta^*_{ss}X$ in K, then $S\beta^*_{ss}X$ in A.

Proof. Let's take a submodule $L \subseteq A$ so that $X + L = A$. In this case, $(X + L) \oplus B = K$. From here $X + (L \oplus B) = K$ can be written. According to Theorem 2.1 (1 ⇔ 3), since $S\beta^*_{ss}X$ in K, it can be written $S + (L \oplus B) = K$. Therefore, there exists the decomposition $(S + L) \oplus B = K$. And since $K = A \oplus B$, $S + L = A$ is found. Similarly, according to Theorem 2.1 (1 ⇔ 3), since $X + W = A$ will be in every submodule $W \subseteq A$ satisfying the equation $S + W = A$. Then we have $S\beta^*_{ss}X$ in A.

It follows from (Wisbauer, 1991) that a submodule $W$of H is said to be *fully invariant* if $g(W) \subseteq W$for each $g \in End(H)$. Elements e_1, e_2 of a ring R is said to be *orthogonal idempotent* if $e_1^2 = 0_R, e_2^2 = 0_R$ and $e_1e_2 = 1_R$.

Theorem 3.8. Let $A = aH$, $B = bH$ and $H = A \oplus B$, where H is a module and the set $\{a, b\}$ is a collection of orthogonal idempotents of $End(H)$. Also, each submodule $X \subseteq H$ can be written as $X = aX + bX$ (especially, X is fully invariant). In this case, module H is Goldie-ss-supplemented iff modules A and B are Goldie-ss-supplemented.

Proof. (⇒) Let H module be Goldie-ss-supplemented. Let's first take an arbitrary submodule $X \subseteq H$. Since the module H is Goldie-ss-supplemented, there is an ss-supplemented submodule S in H such that $X\beta^*_{ss}S$. So there is a $L \subseteq H$ so that $H = S + L$, $S \cap L \subseteq Soc_s\ (L)$. Since $H = S + L$ and $X\beta^*_{ss}S$, then it is obvious that M is $X + L$. Since $L = aL + bL$, it follows the hypothesis that $X + aL + bL$ is H. Since it is known that $X + aL \subseteq A$ and $bL \leq B$, the equations $H = X + aL + B$ and $H = A + bL$ can be written. Also, since $H = A \oplus B$, then $A = X + aL$ and $B = bL$ are found. Again, since $B = bH$, $bS \subseteq bH = bL \subseteq L$ by the hypothesis. From here, $H = S + L = aS + bS + L = aS + L$. Since $aS \subseteq S$ and S is a supplement of L in H, we deduce that aS is S. Therefore, in the module H, $X\beta^*_{ss}aS$. $aS \subseteq A$, $X\beta^*_{ss}aS$ is satisfied in module A due to Lemma 3.7. On the other hand, it is clear that $A = aS + aL$. Here $(aS \cap aL) \ll S = aS$ by (Wisbauer, 1991). By (Kasch, 1982) in 8.1.5, a submodule $aS \cap aL$ is also semisimple. A semisimple submodule aS of A is also an ss-supplemented. Thus, A Goldie-ss-supplemented. Similarly, it can be shown that B is Goldie-ss-supplemented.

(⇐): It is shown that an arbitrary submodule U of the module H, with submodules A and B being Goldie-ss-supplemented. Let's assume that $U_1 = aU$ and $U_2 = bU$. Since $U_1 \subseteq A$ and A are Goldie-ss-supplemented, A has a ss-supplement submodule S_1 with $U_1\beta^*_{ss}S_1$. Then according to Theorem 2.1(1⇔3) there is submodule L_1 such that $L_1 \cap S_1$is semisimple such that $L_1 + S_1 = A$ and $L_1 \cap S_1 \ll S_1$. Since B has a ss-supplement submodule S_2 so that $U_2\beta^*_{ss}S_2$ and $L_2 \subseteq B$ the submodule $L_2 + S_2 = B$ and $L_2 \cap S_2 \ll S_2$ and $L_2 \cap S_2$ semisimple. According to (Gömleksiz & Nişancı Türkmen, 2023) in Proposition 2.6, $U = (U_1 + U_2)\beta^*_{ss}(S_1 + S_2)$ can be written. Moreover, it can be shown that $H = S_1 + S_2 + L_1 + S_2$ and $(S_1 + S_2) \cap (L_1 + L_2$ is equal to $(S_1 \cap L_1) + (S_2 \cap L_2)$. Also, since $L_1 \cap S_1$ is small in S_1 and $L_2 \cap S_2$ is small in S_2. Then we have $(S_1 + S_2) \cap (L_1 + L_2) = (S_1 \cap L_1) + (S_2 \cap L_2) \ll (S_1 + S_2)$ by (Wisbauer, 1991). Since $S_1 \cap L_1$ and $S_2 \cap L_2$ are semisimple submodules, the submodule $(S_1 + S_2) \cap (L_1 + L_2)$ is also a semisimple submodule of $S_1 + S_2$ by (Kasch, 1982). So $S_1 + S_2$ is an ss-supplement of $L_1 + L_2$ in H as a result, H is Goldie-ss-supplemented.

Theorem 3.9. Let H be a Goldie-ss-supplemented module and $X \subseteq H$. Then H/X is Goldie-ss-supplemented.

Proof. Let's take an arbitrary submodule $N/X \subseteq H/X$ with $X \subseteq N \subseteq H$. Since $N \subseteq H$ and H are Goldie-ss-supplemented, H has an ss-supplement submodule S with $N\beta^*_{ss}S$. Let's take any submodule $L/X \subseteq H/X$ that satisfies the sum $N/X + L/X$ is H/X. In this case, there is the equality $N + L = H$ and since $N\beta^*_{ss}S$, then $S + L$ is equal to H according to Theorem 2.1(1⇒ 3). From here, the result $(S + X)/X + L/X = H/X$ is reached. Similarly, the equality $N/X + L/X = H/X$ is provided for each submodule $L/X \subseteq H/X$ providing the sum $(S + X)/X + L/X$ is H/X. It follows from (Gömleksiz & Nişancı Türkmen, 2023) in Proposition 3.1 that $N/X\beta^*_{ss}(S + X)/X$. Thus, $(S + X)/X$ is also an ss-supplement submodule N/X in the factor module H/X. Hence, the H/X is Goldie-ss-supplemented.

Lemma 3.10. Let H be a module and $L \subseteq U \subseteq H$. If the semisimple submodule U of H lifts onto the semisimple submodule L, then $U\beta^*_{ss}L$.

Proof. If U lifts onto the submodule L in M, for every $N \subseteq H$ that satisfies the equation $U + N$ is equal to H. Then the sum $L + N$ is H by (Clark et al., 2006). In addition, since $L \subseteq U$, it is obvious that the sum $U+N$ is H for every $K\subseteq H$ which satisfies the sum $L+K$ is H. Hence by Theorem 2.1. $U\beta_{ss}^{*}L$ is obtained accordingly.

Corollary 3.11. Let H be an ss-lifting module. Then H is Goldie-ss-lifting.

Proof. Let H be ss-lifting. By the definition, there is a decomposition for every submodule A of H with $H=H_1\oplus H_2$ such that $H_1\subseteq A$ and $A\cap H_2\subseteq Soc_s(H)$, from which $Soc_s(H)=Soc(H)\cap Rad(H)$. Since H is ss-lifting each submodule can be lifting on a direct summand. Therefore, each submodule of the module H is equivalent to a direct summand in accordance with Lemma 3.10. So the module H is Goldie-ss-lifting.

Theorem 3.12. Let H be an Goldie-ss-lifting module and X be a submodule of H.Then the factor module $\frac{H}{X}$ is Goldie-ss-lifting also if $\frac{X+K}{X}$ is the direct summand of $\frac{H}{X}$ for each direct summand K of H.

Proof. Let $\frac{N}{X}\leq\frac{H}{X}$. Since H is a Goldie-ss-lifting module, H has direct summand K with $N\beta_{ss}^{*}K$. Since $N\beta_{ss}^{*}K$, then $\frac{N+K}{K}\subseteq Soc_s\left(\frac{H}{K}\right)$ and $\frac{N+K}{N}\subseteq Soc_s\left(\frac{H}{N}\right)$. From here, $\frac{\frac{H}{X}}{\frac{N}{X}}\cong\frac{H}{N}$,$\frac{\frac{N}{X}+\frac{K+X}{X}}{\frac{N}{X}}=\frac{\frac{N+K+X}{X}}{\frac{N}{X}}=\frac{\frac{N+K}{X}+\frac{X}{X}}{\frac{N}{X}}=\frac{\frac{N+K}{X}}{\frac{N}{X}}\cong\frac{N+K}{N}$ and $\frac{\frac{N}{X}+\frac{K+X}{X}}{\frac{K+X}{X}}=\frac{\frac{N+K+X}{X}}{\frac{K+X}{X}}=\frac{\frac{N+K}{X}+\frac{X}{X}}{\frac{K+X}{X}}=\frac{\frac{N+K}{X}}{\frac{K+X}{X}}\cong\frac{N+K}{X+K}\cong\frac{\frac{N+K}{K}}{\frac{X+K}{K}}$. Since $\frac{N+K}{N}$ is small in $\frac{H}{N}$ and $\frac{N+K}{N}$ is semisimple, it is $\frac{\frac{N}{X}+\frac{K+X}{X}}{\frac{N}{X}}\subseteq Soc_s\left(\frac{\frac{H}{X}}{\frac{N}{X}}\right)$. Then $\frac{N+K}{K}\ll\frac{H}{K}$ and $\frac{N+K}{K}$ are semisimple, the factor modules of $\frac{N+K}{K}$, $\frac{\frac{N}{X}+\frac{K+X}{X}}{\frac{X+K}{X}}\ll\frac{\frac{H}{X}}{\frac{X+K}{X}}$ and $\frac{\frac{N}{X}+\frac{K+X}{X}}{\frac{X+K}{X}}$. Thus $\frac{\frac{N}{X}+\frac{K+X}{X}}{\frac{N}{X}}\subseteq Soc_s\left(\frac{\frac{H}{X}}{\frac{N}{X}}\right)$ and $\frac{\frac{N}{X}+\frac{K+X}{X}}{\frac{K+X}{X}}\subseteq Soc_s\left(\frac{\frac{H}{X}}{\frac{K+X}{X}}\right)$ are according to the hypothesis, $\frac{H}{X}$ is a Goldie-ss-lifting module.

Corollary 3.13. Given a Goldie-ss-lifting module H.

(i) If H is distributive, the factor module $\frac{H}{X}$ is Goldie-ss-lifting module for each $X\subseteq H$.

(ii) Let $X\subseteq H$ and each element $e=e^2\in End(M)$ satisfies the condition $e(X)\subseteq X$. Then the factor module $\frac{H}{X}$ is Goldie-ss-lifting.

Proof. (i) Since H is a Goldie-ss-lifting module, M has a direct summand D with $X\beta_{ss}^{*}D$. Since $X\beta_{ss}^{*}D$, it is $\frac{X+D}{D}\subseteq Soc_s\left(\frac{H}{D}\right)$ and $\frac{X+D}{X}\subseteq Soc_s\left(\frac{H}{X}\right)$. Here there is a submodule $D'\subseteq H$ with $H=D\oplus D'$. Then the equation $\frac{H}{X}=\frac{D+X}{X}+\frac{D'+X}{X}$ can be written. Since M is distributive, then $X=X+(D\cap D')=(X+D)\cap(X+D')$ is obtained. So the direct sum $\frac{(D+X)}{X}\oplus\frac{(D'+X)}{X}$ is $\frac{H}{X}$. For each direct summand D of H, $\frac{D+X}{X}$ is a direct summand of $\frac{H}{X}$. By Theorem 3.12, the factor module $\frac{H}{X}$ is Goldie-ss-lifting.

(ii) Let D be an arbitrary direct summand of H. Let's consider the canonical projection $e:H\to D$. In this case $e^2=e\in End(H)$. Since $e(X)\subseteq X$ in the hypothesis, $e(X)=X\cap D$ is satisfies. Also, since D is a direct summand of H, we can write $H=D\oplus D'$ for some $D'\subseteq H$. That is, the equation $X=(X\cap D)\oplus(X\cap D')$ can be written. Then we have $\frac{D+X}{X}=\frac{[D\oplus(X\cap D')]}{X}$ and $\frac{(D'+X)}{X}=\frac{[D'\oplus(X\cap D)]}{X}$

are obtained. Therefore, $\frac{H}{X} = \frac{(D+X)}{X} + \frac{(D'+X)}{X} = \frac{[D\oplus(X\cap D')]}{X} + \frac{(D'+X)}{X}$ is found. It can be shown from $\big(D\oplus(X \cap D')\big) \cap (\,D' + X) = X$ that $\frac{H}{X} = \frac{[D\oplus(X\cap D')]}{X} \oplus \frac{(D'+X)}{X}$. By Theorem3.12 the factor module $\frac{H}{X}$ is Goldie-ss-lifting, as required.

Proposition 3.14 Let H be a Goldie-ss-lifting module. Then H is a Goldie*-lifting module. If $Soc_s(H) \ll H$, then the converse holds.

Proof. Let H be a Goldie-ss-lifting module. $H = D\oplus D'$ so that $(N + D)/N \subseteq Soc_s(H/N)$ and $(N + D)/D \subseteq Soc_s(H/d)$ for any submodule N of H. Therefore $(N + D)/N \ll H/N$ and $(N + D)/D \ll H/D$. So H is a Goldie*-lifting module. Conversely, let $N \subseteq H$. By assumption, H has a decomposition $H = D\oplus D'$ so that $(N + D)/N \ll H/N$ and $(N + D)/D \ll H/D$. Then $H = D + D' = N + D'$ and $(N + D)/D \subseteq (Soc_s(H) + D)/D$. Let $\theta : (D + D')/D \to D'$, $\psi : D'/(N \cap D') \to (N + D')/N$ be isomorphisms and $f: D' \to D'/(N \cap D')$ be an epimorphism. Set $h = \psi f \theta$. By a similar argument to (Gömleksiz & Nişancı Türkmen, 2023) in Proposition 2.6, $(N + D)/N = h((N + D)/D)$. Since $(N + D)/D \subseteq (Soc_s(H) + D)/D$, we have $(N + D)/N \subseteq h(Soc_s(H) + D)/N$. Hence, H is a Goldie-ss-lifting module.

4. CONCLUSION

In this paper, we have considered the Goldie-ss-lifting modules as a specialized notion of Goldie*-lifting modules. In particular, it is obtained fundamental module structure theorems by the help of a relation β^*_{ss} that is more special than β^* .

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