1-TYPE AND HARMONIC 1-TYPE TIMELIKE CURVES IN SEMI-EUCLIDEAN SPACE R_2^4

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Geliş Tarihi / Received: 31.08.2021 *Araştırma Makalesi/Research Article*
Kabul Tarihi / Accepted: 20.09.2021 *DOI: 10.38065/euroasiaorg.709*

ABSTRACT

In the present study we consider 1-type, biharmonic, weak biharmonic and harmonic 1-type curves in semi-Euclidean space R_2^4 according to the timelike Frenet frame. We give some characterizations and classifications of these type curves.

Keywords: Biharmonic timelike curve, 1-type timelike curve, Weak biharmonic timelike curve, Harmonic 1-type timelike curve, Semi-Euclidean space.

2020 Mathematics Subject Classification: *53A04, 53A40.*

1. INTRODUCTION

Chen and Ishikawa classified biharmonic curves in pseudo-Euclidean space E_v^n [1]. They showed that every biharmonic curve lies in a 3-dimensional totally geodesic subspace. Further, Inoguchi gave a classification of biharmonic curves in semi-Euclidean 3-space. He pointed out that every biharmonic Frenet curve in Minkowski 3-space E_1^3 is a helix whose curvature and torsion satisfy $\tau^2 = \kappa^2$ [5],[6]. Kılıç examined finite type curves and surfaces and gave a classification of harmonic and weak biharmonic surfaces [7].

In this paper we give the characterizations of 1-type, biharmonic, weak biharmonic, harmonic 1-type curves with the help of the Frenet frame components presented for timelike curves in semi-Euclidean space R_2^4. Also, we present some classifications according to the curvatures of such curves.

2. PRELIMINARIES

To meet the requirements in the next sections, the basic elements of the theory of curves in the semi Euclidean space R_2^4 are briefly presented in this section. A more complete elementary information can be found in [10]. Semi-Euclidean space R_2^4 is an Euclidean space provided with standard flat metric given by

$$g = -dx_1^2 - dx_2^2 + dx_3^2 + dx_4^2$$

where (x_1, x_2, x_3, x_4) is a rectangular coordinate system of the R_2^4. A vector v in R_2^4 is called a spacelike, timelike or null (lightlike) if respectively hold $g(v, v) > 0$, $g(v, v) < 0$ or $g(v, v) = 0$ and $v \neq 0$. The norm of a vector v is given by $\|v\| = \sqrt{|g(v,v)|}$. Therefore, v is a unit vector if $g(v, v) = \pm 1$. Two vectors v and w are said to be orthogonal if $g(v,w) = 0$ [10].

Similarly, an arbitrary curve $\gamma = \gamma(s)$ in R_2^4 can locally be spacelike, timelike or null (lightlike) if all of its velocity vectors $\gamma'(s)$ are, respectively, spacelike, timelike or null (lightlike). The velocity of the curve γ is given by $\|\gamma'(s)\|$. Thus, a timelike curve γ is said to be parametrized by arc length function *s* if $g(\gamma', \gamma') = -1$ [10].

Let $\{T(s), N(s), B_1(s), B_2(s)\}$ denotes the moving Frenet frame along in the semi-Euclidean space R_2^4, then $T(s)$, $N(s)$, $B_1(s)$ and $B_2(s)$ are called the tangent, the principal normal, the first binormal, and the second binormal vector fields of γ, respectively.

A unit speed curve γ is said to be a Frenet curve if $g(\gamma'', \gamma'') \neq 0$. Let γ be a C^∞ special timelike Frenet curve with timelike principal normal, spacelike first binormal and second binormal vector fields in R_2^4, parametrized by arc length function *s*. Moreover, non-zero C^∞ scalar functions κ_1, κ_2 and κ_3 be the first, second, and third curvatures of γ, respectively. Then for the C^∞ special timelike Frenet curve γ, the following Frenet formula is given by

(2.1) $$\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & -\kappa_1 & 0 & 0 \\ \kappa_1 & 0 & \kappa_2 & 0 \\ 0 & \kappa_2 & 0 & \kappa_3 \\ 0 & 0 & -\kappa_3 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix}$$

where T, N, B_1 and B_2 mutually orthogonal vector fields satisfying $g(T, T) = g(N,N) = -1$, $g(B_1,B_1) = g(B_2,B_2) = 1$. (for the semi-Euclidean space E_v^{n+1}, see [3],[9]).

Let $\gamma = \gamma(s): I \subset R \to R_2^4$ be an arclengthed curve. Namely the velocity vector field γ' satisfies $g(\gamma', \gamma') = 1$. A unit speed curve γ is said to be a geodesic if $\nabla_{\gamma'}\gamma' = 0$, where ∇ is the Levi-Civita connection of the R_2^4. In particular an arclengthed curve γ is said to be a geodesic if $\kappa_1 = 0$, where is the curvature of γ. Note that if $\kappa_1 = 0$ then automatically $\kappa_2 = 0$ and $\kappa_3 = 0$.

Let us denote the Laplace-Beltrami operator by Δ of γ and the mean curvature vector field along γ by *H*. The Frenet-Serret formulae of γ imply that the mean curvature vector field *H* is given by

(2.2) $$H = \nabla_{\gamma'}\gamma' = \nabla_{\gamma'}T = -\kappa_1 N.$$

The Laplacian operator of γ is defined by

(2.3) $$\Delta = -\nabla_{\gamma'}^2 = -\nabla_{\gamma'}\nabla_{\gamma'}$$

([1],[4]).

The Laplacian operator along γ associated the connection in the normal bundle is defined by

$$\Delta^\perp H = -\nabla_{\gamma'}\nabla_{\gamma'}H.$$

Definition 2.1. A unit speed timelike curve $\gamma: I \to R_2^4$ is said to be 1-type timelike curve if

$$\Delta H = \lambda H.$$

Definition 2.2. A unit speed timelike curve $\gamma: I \to R_2^4$ is said to be biharmonic timelike curve if

$$\Delta H = 0.$$

Lemma 2.1. *The mean curvature vector field H is harmonic* ($\Delta H = 0$) *if and only if*

$$\nabla_{\gamma'}\nabla_{\gamma'}\nabla_{\gamma'}\gamma' = 0$$

(see [1]). It is clear that in this case the curve will be biharmonic curve.

Theorem 2.1. *If M is the Euclidean space* E^m *, then along the curve* γ*, H satisfies* $\Delta H = 0$ *if and only if* γ *is biharmonic (i.e.,* $\Delta(\Delta\gamma) = 0$ *since* $\Delta H = 0$*)* [2].

Definition 2.3. A unit speed timelike curve $\gamma : I \to R_2^4$ is said to be harmonic 1-type timelike curve if

$$\Delta^{\perp} H = \lambda H.$$

Definition 2.4. A unit speed timelike curve $\gamma : I \to R_2^4$ is said to be weak biharmonic timelike curve if

$$\Delta^{\perp} H = 0.$$

3. 1-TYPE AND BIHARMONIC TIMELIKE CURVES

Theorem 3.1. *Let* γ *be an arclengthed parametrized timelike Frenet curve of the* R_2^4*. Then, along the curve* γ*,* $\Delta H = \lambda H$ *holds if and only if*

$$\begin{aligned} &\kappa_1\kappa_1' = 0 \\ &\kappa_1'' - \kappa_1^3 + \kappa_1\kappa_2^2 = -\lambda\kappa_1 \\ &2\kappa_1'\kappa_2 + \kappa_1\kappa_2' = 0 \\ &\kappa_1\kappa_2\kappa_3 = 0. \end{aligned} \tag{3.1}$$

Proof. From the equalities (2.1), (2.2) and (2.3) we get

$$\Delta H = 3\kappa_1\kappa_1' T + (\kappa_1'' - \kappa_1^3 + \kappa_1\kappa_2^2)N + (2\kappa_1'\kappa_2 + \kappa_1\kappa_2')B_1 + \kappa_1\kappa_2\kappa_3 B_2.$$

By (2.2) and $\Delta H = \lambda H$ we have

$$3\kappa_1\kappa_1' T + (\kappa_1'' - \kappa_1^3 + \kappa_1\kappa_2^2)N + (2\kappa_1'\kappa_2 + \kappa_1\kappa_2')B_1 + \kappa_1\kappa_2\kappa_3 B_2 = -\lambda\kappa_1 N.$$

Thus, the equations (3.1) are obtained. Conversely, the equations (3.1) satisfy the equality $\Delta H = \lambda H$.

Theorem 3.2. *Let* γ *be an arclengthed parametrized timelike Frenet curve of the* R_2^4*. Then, along the curve ,* $\Delta H = \lambda H$ *holds if and only if* γ *is a circular helix in a three-dimensional subspace of the* R_2^4*, where*

$$\lambda = \kappa_1^2 - \kappa_2^2. \tag{3.2}$$

Proof. From theorem (3.1) we have the equations (3.1). Since γ is a timelike Frenet curve, $\kappa_1 \neq 0.$ Thus the equation (3.1) shows that κ_1 and κ_2 are constants and $\kappa_3 = 0$. Thus γ is a circular helix in a three-dimensional subspace of the R_2^4 and we obtain the equation (3.2).

Conversely, since γ is a circular helix in a three-dimensional subspace of the R_2^4,

$\lambda = \kappa_1^2 - \kappa_2^2$, κ_1 and κ_2 are non-zero constants and $\kappa_3 = 0$. From that $\Delta H = \lambda H$ is satisfied.

Corollary 3.1. *An arclengthed parametrized timelike1-type curve* $\gamma : I \to R_2^4$ *is a circular helix in* R_2^3*, for* $\lambda = \kappa_1^2 - \kappa_2^2$.

Theorem 3.3. *Let be an arclengthed timelike curve in* R_2^4. *Then, along the curve* γ, $\Delta H = 0$ *holds if and only if* γ *is a geodesic or a circular helix (*$\kappa_1 = \pm\kappa_2$*) in a three-dimensional subspace of the* R_2^4.

Proof. Let I be an open interval and $\gamma : I \to R_2^4$ be an arclengthed timelike curve where its arclength is s. Let $\{T(s), N(s), B_1(s), B_2(s)\}$ be the timelike Frenet frame field of γ. By (2.2), direct computation shows that

$$\nabla_{\gamma'} H = \kappa_1^2 T + \kappa_1' N + \kappa_1\kappa_2 B_1.$$

Let us compute the Laplacian of H

$$\begin{aligned}\Delta H &= -\nabla_{\gamma'}\nabla_{\gamma'} H \\ &= 3\kappa_1\kappa_1' T + (\kappa_1'' - \kappa_1^3 + \kappa_1\kappa_2^2)N + (2\kappa_1'\kappa_2 + \kappa_1\kappa_2')B_1 + \kappa_1\kappa_2\kappa_3 B_2.\end{aligned}$$

Thus along the curve γ, $\Delta H = 0$ holds if and only if $\kappa_1 = 0$ or κ_1, κ_2 are non-zero constants and $\kappa_3 = 0$. So, γ is either a geodesic in the same space or a circular helix in a three-dimensional subspace of the R_2^4.

Conversely, every geodesic curve satisfies $\Delta H = 0$, since $\kappa_1 = 0$. Similarly, every circular helix in a three-dimensional subspace of the R_2^4 satisfies $\Delta H = 0$, since κ_1, κ_2 are non-zero constants and $\kappa_3 = 0$.

Corollary 3.2. *An arclengthed parametrized biharmonic timelike curve* $\gamma : I \to R_2^4$ *is either a geodesic in the same space or a circular helix (*$\kappa_1 = \pm\kappa_2$*) in* R_2^3.

4. HARMONIC 1-TYPE AND WEAK BIHARMONIC TIMELIKE CURVES

Let's denote the normal bundle of the timelike curve $\gamma : I \subset R \to R_2^4$ with $\chi^\perp(\gamma(I))$, where $\chi^\perp(\gamma(I))$ = S_P $\{N(s), B_1(s), B_2(s)\}$. For $\forall X \in \chi^\perp(\gamma(I))$, the normal connection $\nabla^\perp$ and the normal Laplace-Beltrami operator $\Delta^\perp$ are defined as

(4.1)
$$\begin{aligned}\nabla_T^\perp X &= \nabla_T X + \langle \nabla_T X, T\rangle T \\ \Delta_T^\perp X &= -\nabla_T^\perp \nabla_T^\perp X.\end{aligned}$$

Theorem 4.1. *Let* γ *be an arclengthed parametrized timelike Frenet curve of the* R_2^4. *The curve* γ *is a harmonic 1-type timelike curve if and only if*

(4.2)
$$\begin{aligned}\kappa_1'' + \kappa_1\kappa_2^2 &= -\lambda\kappa_1 \\ 2\kappa_1'\kappa_2 + \kappa_1\kappa_2' &= 0 \\ \kappa_1\kappa_2\kappa_3 &= 0.\end{aligned}$$

Proof. Since the curve γ is a harmonic 1-type timelike curve, along the curve γ, the equality $\Delta^\perp H = \lambda H$ is provided. From the (4.1),

(4.3)
$$\begin{aligned}\nabla_{\gamma'}^\perp H &= \kappa_1' N + \kappa_1\kappa_2 B_1 \\ \Delta^\perp H &= (\kappa_1'' + \kappa_1\kappa_2^2)N + (2\kappa_1'\kappa_2 + \kappa_1\kappa_2')B_1 + \kappa_1\kappa_2\kappa_3 B_2.\end{aligned}$$

Thus, the equalities (4.2) are obtained.

Theorem 4.2. *Let* γ *be a harmonic 1-type timelike curve in the* R_2^4 *and let s be its arclength function. Then:*

i) γ is a straight line,

ii) γ is a curve with the curvature $\kappa_1 = c_1 \sin(\sqrt{\lambda} s) + c_2 \cos(\sqrt{\lambda} s)$,

iii) γ is a a helix in the R_2^3 with the curvatures κ_1=constant, $\kappa_2 = \pm\sqrt{\lambda}$,

iv) γ is a curve in the R_2^3 with the non-constant curvatures and

$$\kappa_2 = \frac{c}{\kappa_1^2}$$

where c, c_1, c_2 are integral constants.

Proof. If we assume that $\kappa_1 = 0$, then the equations (4.2) are satisfied. So, γ is a straight line. If the κ_1 is a non-constant function and $\kappa_2 = 0$, then from the equations (4.2), we find $\kappa_1'' + \lambda\kappa_1 = 0$ and the solution of differential equation is

$$\kappa_1 = c_1 \sin(\sqrt{\lambda} s) + c_2 \cos(\sqrt{\lambda} s).$$

If κ_1 and κ_2 are non-zero constant, then from the equations (4.2), we find $\kappa_3 = 0$ and $\kappa_2 = \pm\sqrt{\lambda}$. If we take κ_1 and κ_2 are non-constant functions, then from the equations (4.2), we find $\kappa_3 = 0$, $\kappa_2 = \dfrac{c}{\kappa_1^2}$ and κ_1 can be calculated if required.

Conversely, if γ is a curve given with arc-length parameter s and if one of (i); (ii); (iii) or (iv) holds, then γ is the harmonic 1-type timelike curve.

Theorem 4.3. *Let* γ *be an arclengthed parametrized timelike curve of the* R_2^4. *The curve* γ *is a weak biharmonic timelike curve if and only if*

$$\begin{aligned} &\kappa_1'' + \kappa_1\kappa_2^2 = 0 \\ &2\kappa_1'\kappa_2 + \kappa_1\kappa_2' = 0 \qquad (4.4) \\ &\kappa_1\kappa_2\kappa_3 = 0. \end{aligned}$$

Proof. The proof is provided with the equalities (4.4) and $\Delta^{\perp} H = 0$.

Corollary 4.1. *Every straight line of the* R_2^4 *is a weak biharmonic timelike curve.*

Theorem 4.4. *Let* γ *be a weak biharmonic timelike curve in the* R_2^4 *and let s be its arclength function. Then:*

i) γ is a straight line,

ii) γ is a pseudo circle,

iii) γ is a cornu spiral with $\kappa_1 = c_1 s + c_2$, where c_1, c_2 are integral constants

iv) γ is a curve of the R_2^3 with the non-constant curvatures, in which case

(4.5)
$$\kappa_3 = 0$$
$$\kappa_2 = \frac{c}{\kappa_1^2}$$
$$\kappa_1'' + \frac{c}{\kappa_1^3} = 0$$

Proof. If we assume that $\kappa_1 = 0$, then the equations (4.4) are satisfied. So, γ is a straight line. If κ_1 is a non-zero constant and $\kappa_2 = 0$, then the equations (4.4) are also satisfied. So, γ is a pseudo circle. Further, If κ_1 is a non-constant function and $\kappa_2 = 0$, then from the equations (4.4), we find $\kappa_1'' = 0$ and the solution of differential equation is $\kappa_1 = c_1 s + c_2$. So, γ is a cornu spiral. Finally, if κ_1 and κ_2 are non-constant functions, then from the equations (4.4), we find the equalities (4.5).

Conversely, if is a curve given with arc-length parameter *s* and if one of (i); (ii); (iii) or (iv) holds, then γ is weak biharmonic timelike curve.

5. CONCLUSION

In this paper we gave the characterizations of 1-type, biharmonic, weak biharmonic, harmonic 1-type timelike curves in semi-Euclidean space R_2^4 in terms of curvatures. More precisely, we showed that every 1-type timelike curve in R_2^4 is a circular helix in a three-dimensional subspace of the R_2^4 ,for $\lambda = \kappa_1^2 - \kappa_2^2$ and every biharmonic timelike curve is either a geodesic in the same space or circular helix $(\kappa_1 = \pm\kappa_2)$ in a three-dimensional subspace of the R_2^4**.** Also, we gave some classifications for weak biharmonic and harmonic 1-type timelike curves in the R_2^4.

REFERENCES

[1] B. Y. Chen and S. Ishikawa, *Biharmonic surfaces in pseudo-Euclidean spaces*, Mem. Fac. Sci. Kyushu Univ. Ser. A **45** (1991), 323-347.

[2] B. Y. Chen, *On the total curvature of immersed manifolds, VI : Submanifolds of finite type and their applications*, Bull. Ins. Math. Acad. Sinica **11** (1983), 309-328.

[3] A. C. Çöken, and A. Görgülü, *On the Joachimsthal's Theorems in semi Euclidean Spaces*, Nonlinear Analysis: Theory, Methods and Applications, **70** (2009), 3932-3942.

[4] A. Ferrandez, P. Lucas and M. A. Merono, *Biharmonic Hopf cylinders*, Rocky Mountain J. **28** (1998), 957-975.

[5] J. Inoguchi, *Biharmonic curves in Minkowski 3-space*, International J. Math. Sci. **21** (2003), 1365-1368.

[6] J. Inoguchi, *Biharmonic curves in Minkowski 3-space part II*, International J. of Math. and Mathematical Sci. (2006), Article ID 92349, 1-4.

[7] B. Kilic¸*Finite type curves and surfaces*, Ph. Thesis, University of Hacettepe, 2002.

[8] H. Kocayigit, H. Hacısalihoglu, *1-Type and Biharmonic Curves in Euclidean 3-Space*, International Electronic Journal of Geometry , **4** (2011), 97-101.

[9] S. Ersoy, M. Tosun, *Timelike Bertrand Curves in Semi-Euclidean Space*, arXiv:1003.1220v1[math.DG].

[10] B. O'Neill, *Semi-Riemannian Geometry with Applications to Relativity*, Academic Press, New York, 1983.

[11] G. Ozturk, B. K. Bayram, K. Arslan, *Weak Biharmonic Curve and Harmonic 1-Type Curve in Semi-Euclidean space* E_1^4, Acta Universitatis Apulensis, **40** (2014), 97-101.